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IDENTIFICATION OF DYNAMIC STIFFNESS OF BALL BEARING JOINTS USING SENSITIVITY ANALYSIS

Wan-Tack Rim*, Kwang-Joon Kim* and Chong-Won Lee**

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Natural frequencies are used to identify the joint stiffness of a shaft-bearing system, where the shaft is assumed as a Rayleigh beam and the bearing as a linear spring. The eigenvalues for the case of uniform circular shaft are derived in a rigorous manner in terms of nondimensionalized parameters for the location of beamgs, bearing stiffness, and slenderness of the shaft. Then, sensitivities of the eigenvalues are calculated with respect to the stiffness ratio of shaft and bearing and the bearing location. Based upon these sensitivity charts, an iterative procedure is proposed to identify the bearing stiffness in an optimum sense. The step by step procedure is illustrated through a case study for a simple shaft-bearing system, where the stiffness of a set of ball bearings is successfully identified.

Key Words: Sensitivity, Stiffness Parameter, Location Parameter

1. INTRODUCTION

It is often necessary to predict the dynamic behavior of a mechanical system at the design stage in order that the system may be operated in dynamically favorable conditions once it has been manufactured. So far dynamic analysis of solid mechanical components has been very successful owing to advanced computer softwares, e.g., FEM programs, while that of a whole system has not been mainly due to the lack of information for the dynamic properties on mechanical joints constructed by bearings, bolts and welding. In this aspect, hence, many articles have been published for the last decade with regard to the identification of the dynamic properties of the above mentioned mechanical joints (Tlusty and Moriwaki, 1976 ; Yoshimura and Okushima, 1977 ; Choi, 1987).

The general practice for the identification of bearing joints was to express a system with bearing joints by a mathematical model and to estimate the joint properties somehow by comparing the predictions of the dynamic properties/behavior of the whole system with the experimental measurements (Stone, 1982). The comparison was based upon modal parameters or forced responses or any other properties. The modal parameters here may mean natural frequencies only or may include damping ratios and mode shape vectors. Although it is generally agreed that more experimental measurements will make the parameter estimation more reliable, its effectiveness is surely dependent upon the accuracy of the measured results. Admitting that accurate estimation of natural frequencies is far easier than of damping ratios or mode shapes in the experimental method, and considering that natural frequencies are more directly coupled to the critical speed of the rotor-bearing system. the use of the measured natural frequencies alone, not together with the other modal paramentrs, can be justified by arranging the test set-up in such a way that the natural frequencies are most sensitive to the bearing stiffness.

In this paper a stationary shaft-bearing system is analyzed by assuming that the shaft be a Rayleigh beam and the bearing act as a linear spring. The eigenvalues are derived in a rigorous manner for the case of uniform circular shaft in terms of nondimensionalized parameters for the location of bearings, bearing stiffness, and slenderness of the shaft. Subsequently sensitivities of the eigenvalues are calculated with respect to (i) the stiffness ratio of shaft and bearing and (ii) the bearig location. Based upon these sensitivity charts and iterative procedure is proposed to identify the bearing stiffness in an optimum sense. The step by step procedure will be illustrated through a case study for a simple shaft-bearing system, where the stiffness of a set of ball bearings is successfully estimated.

2. FORMULATION OF ELGNVALUE PROBLEMS

Although the case study in the later part of this paper will be made on a stationary system because of difficulties in experimentation, in this section, the eigenvalue problem is formulated for a rotating system which is closer to a real situation.

The equations of motion of a rotating shaft-bearing system, where the shaft is assumed to be a Rayleigh beam and each bearing support a linear spring, are given by :

$$\rho A(Z)\frac{\partial^2 U}{\partial_t{}^2}-\frac{\partial}{\partial Z}\left\{J_t(Z)\frac{\partial}{\partial Z}\left(\frac{\partial^2 U}{\partial t^2}\right)\right\}+j\Omega\frac{\partial}{\partial Z}\left\{J_p(Z)\frac{\partial}{\partial Z}\left(\frac{\partial U}{\partial t}\right)\right\}+\frac{\partial^2}{\partial Z^2}\left\{EI(Z)\frac{\partial^2 U}{\partial Z^2}\right\}=P(Z,\ t) \quad (1)$$

*Department of Production Engineering
**Department of Mechanical Engineering, Korea Advanced of Science and Technology P.O.Box 150 Cheongryang Seoul, 130 ~650, Korea

with constraint conditions at each bearing support as below :

$$\left.\frac{\partial^3 U}{\partial Z^3}\right|_{Z=Z_i+\exists}=\left.\frac{\partial^3 U}{\partial Z^3}\right|_{Z=Z_i+\exists}+K_i U(Z,\quad t)|_{Z=Z_i,}\quad {}^{i=1,2,\cdots,n} \tag{2}$$

where

$U(Z,\ t)=X(Z,\ t)+jY(Z,\ t)$: Deformation of the shaft

$P(Z,t)$: External force

$A(Z,\ t)$: Cross-sectional area of the shaft

ρ : Density of the shaft material

$J_t(Z)$: Diametric mass moment of inertia

$J_p(Z)$: Polar mass moment of inertia

$EI(Z)$: Flexural stifffness of the shaft

$\Omega(Z)$: Rotating speed of the shaft

n : Number of bearing support

Z_i : Location of the i-th bearing support

K_i : Stiffness of the i-th bearing support

$\exists$: Small distance ($\ll Z_i$)

To make the analysis simple, the model as shown in Fig. 1 is considered here, where the swhaft with uniform cross-sectional area is supported symmetrically by two identical bearings and there are no external force acting on the shaft. The above Eqs.(1, 2) then can be re-expressed in nondimensionalized form as below :

$$\frac{\partial^2 u}{\partial^2 \tau}+\frac{\partial^4 u}{\partial z^4}-r^2\frac{\partial^4 u}{\partial z^2\partial\tau^2}+jw\frac{\partial^3 u}{\partial z^2\partial\tau}=0 \tag{3}$$

$$\left.\frac{\partial^3 u}{\partial z^3}\right|_{Z=Z_i-\epsilon}=\left.\frac{\partial^3 u}{\partial z^3}\right|_{Z=Z_i+\epsilon}+Ku_i \tag{4}$$

in terms of nondimensional papameters as follows :

$$\begin{array}{lll} u=U/\ell & z=Z/\ell & \tau=t/\{m\ell^4/EI\}^{1/2} \\ w^2=\rho R^2\Omega^2/E & k=K\ell^3/EI & r=R/2\ell \end{array} \tag{5}$$

where k denotes the stiffness ratio of the bearing and shaft, r the slenderness ratio of the shaft, ϵ a small number ($\ll 1$), and $i=1,\ 2$.

Solutions of Eqs.(3, 4) for free boundary conditions at both ends can be easily obtained by dividing the system into theree sections(Hong, 1989) :

$$0\leq z\leq z_1,\quad z_1\leq z\leq z_2, \text{ and } z_2\leq z\leq 1.0$$

and by imposing the following conditions :

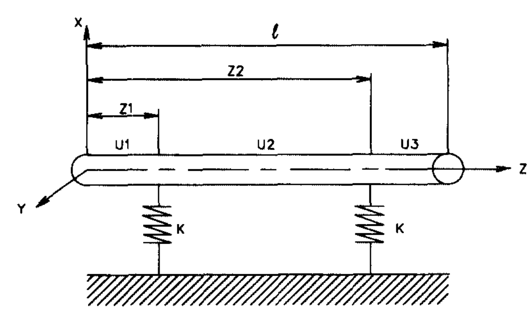

Fig. 1 Idealized model of a shaft-bering system

$$\begin{array}{lll} \text{at } z=0 & ; & u_1''=0, -r^2\ddot{u}_1'+u_1'''+jw\dot{u}_1''=0 \\ \text{at } z=z_1 & ; & u_1=u_2, u_1'=u_2',\quad u_1''=u_2'' \\ & & u_1'''=u_2'''+ku_2 \\ \text{at } z=z_2 & ; & u_2=u_3,\ u_2'=u_3',\quad u_2''=u_3'' \\ & & u_2'''=u_3'''+ku_3 \\ \text{at } z=1 & ; & u_3''=0, -r^2\ddot{u}_3'+u_3'''+jw\dot{u}_3'' \end{array} \tag{6}$$

on the solution of Eq.(3) which is assumed to take following form :

$$u_i(z,\ \tau)=V_i e^{\lambda Z}e^{\nu\tau},\quad i=\begin{cases} 1 \text{ for } 0\leq z\leq z_1 \\ 2 \text{ for } z_1\leq z\leq z_2 \\ 3 \text{ for } z_2\leq z\leq 1. \end{cases} \tag{7}$$

Substitution of Eq.(7) into the Eq.(3) yields a characteristic equation as follows :

$$\lambda^4+(r^2\nu^2-\omega\nu)\lambda^2-\nu^2=0 \tag{8}$$

from which the sapcial characteristic value λ can be derived in terms of the other parameters as follows.

$$\lambda=\pm\Lambda_1,\quad \pm\Lambda_2\geq 0 \tag{9}$$

whereo both Λ_1 and Λ_2 are positive and given respectively by :

$$\begin{array}{l} \Lambda_1=[(r^2\nu^2-\omega\nu)/2+\{(r^2\nu^2-\omega\nu)^2/4+\nu^2\}^{1/2}]^{1/2} \\ \Lambda_2=[-(r^2\nu^2-\omega\nu)/2+\{(r^2\nu^2-\omega\nu)^2/4+\nu^2\}^{1/2}]^{1/2} \end{array} \tag{10}$$

By substituting the Eqs. (7, 9, 10) into the boundary and constratint conditions given by the Eq. (6), an eigenvalue problem can be formulated as follows :

$$[C]\{A\}=\{0\} \tag{11}$$

where $[C]$is a 12×12 matrix constructed with the characterstic parameters and $\{A\}$ a constant 12×1 column vector which is related to the constant coefficients $V_i's$ in the Eq. (7). Details of this are explained in the APPENDIX. The eigenvalues, hence, can be obtained by solving the following equation :

$$\mathrm{Det}[C]=0 \tag{12}$$

3. SENSITIVITY ANALYSIS

Now the sensitivity of the eigenvalue with respect to the two parameters, i.e., stiffness ratio of the shaft and bearing, defined as k in Eq.(5), and bearing location, defined by :

$$\alpha=z_2-z_1 \tag{13}$$

is discussed. There are several methods to determine the sensitivity (Fox and Kappor, 1968 ; Adelman and haftka, 1986), one of which, a numerical method, was used in this study to calculate the mean change rate of the eigenvalue to each parameter.

Figure 2(a) shows the variation of the eigenvalue ν with respect to k and α for the case of the stationary shaft-bearing system, i.e., $\omega=0$, from which the sensitivity of the eigenvalue was calculated as shown in Figs. 2(b) and 2(c). Fig. 2(b) indicates

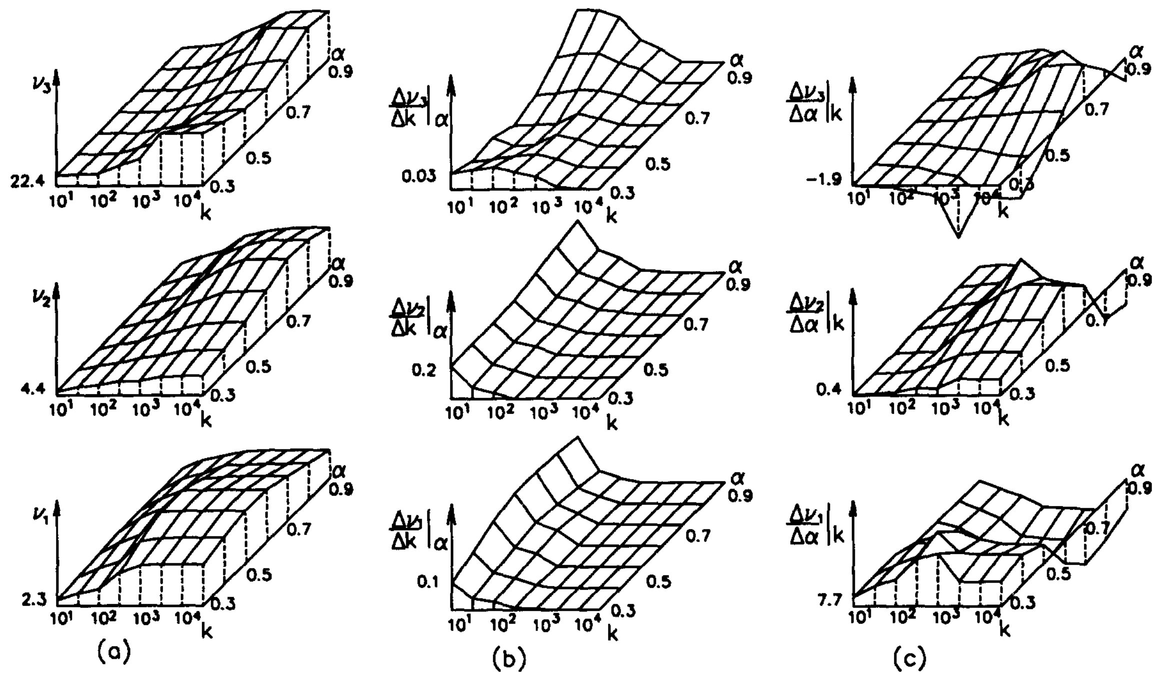

Fig. 2 (a) Eignvalues when α and k varies, (b) Sensitivity of eigenvalues with respect to stiffness parameter k, (c) Sensitivity of eigenvaiues with respect to location parameter α

that the sensitivity is relatively high when the stiffness ratio is low(10 to 100) regardless of the location parameter for the first mode and that the situations for the first mode and that the situations for the second and third mode are similar. Fig 2c, however, shows that the sensitivity with respect to the location parameter takes rather complicated shapes depending upon the stiffness ratio and the mode.

It is desirable from the practical point of view that the sensitivity with respect to the stiffness ratio be maximum while the one with respect to the bearing location be minimum because the assumption of the point contact at the bearing support may not be true and the exact location of the point contact is very difficult. By considerig this point, a systematic procedure to estimate the stiffness of the bearing support by the measurement of natural frequencies is suggested as follows and the flow chart is shown in Fig. 3.

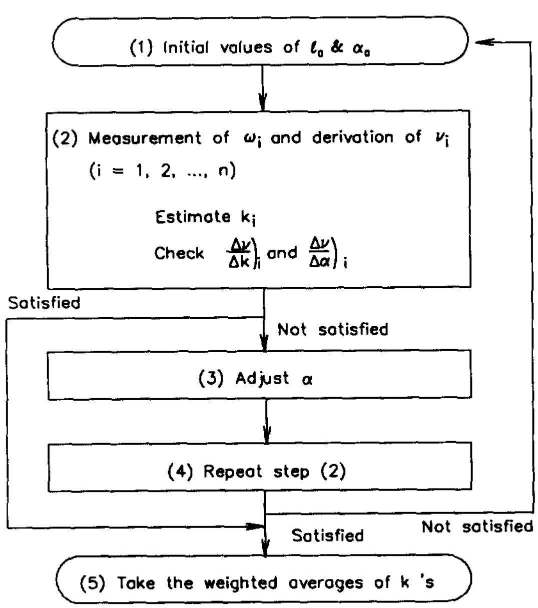

Fig. 3 Process flow for searching bearing stiffness

(1) A shaft of a given dimension is installed symmetrically onto two identical bearings as shown in Fig. 1 for a chosen value of the location parameter.

(2) Natural frequencies are measured from experiments and nondimensional eigenvalues are derved using the Eq.(5). The stiffness parameters are subsequently obtained using the chart as shown in Fig. 2(a) for every mode. Then sensitivities of each eigenvalue with respect to the stiffness and location parameters are investigated. If all of the following conditions are satisfied.

- small variation in k_i
- high sensitivity with respect to k_i
- low sensitivity with respect to α_i

then go to step (5). Otherwise, go to step(3).

(3) Adjust the location parameter α in such a way that a weighted sum of the sensitivity with respect to α for each mode may be minimized.

(4) Repeat step(2). If satisfied, go to step(5). If not satisfied, then change the shaft length and go to step(1).

(5) Take the weighted average of the stiffness parameters by considering the sensitivities i-th respect to the stiffness and location parameters as follows:

$$k=\sum_i^n W_i k_i$$

where

$$W_i=\frac{S_{ki}/|S_{\alpha i}|}{\sum_j^n S_{kj}/|S_{\alpha j}|}$$

Table 1 Identified joint stiffness at $\ell=750$mm, $\alpha=0.6$

Natural frequency	Eigen value	Stiffness parameter	Bearing stiffness	$\Delta\nu/\Delta k$	$\Delta\nu/\Delta k$	W_i
241 Hz	21.87	10000	2.0×10^{8}	1.1×10^{-6}	-18.31	0.002
478 Hz	44.25	4380	8.7×10^{7}	4.1×10^{-4}	133.47	0.118
691 Hz	62.87	6220	1.2×10^{8}	1.8×10^{-3}	78.98	0.879
Weighted stiffness parameter 6010						
Weighted bearing stiffness 1.2×10^{8} N/m						

4. A CASE STUDY

Although the procedure in the previous section is rigorous and systematic, decision making is so subjective. Hence, in this case study, the decision was made to some extent based on engineering senses.

A shaft of diameter 30mm and length 750mm was installed onto a pair of ball bearings of width 30mm. Based on some preliminary measurements and estimations, $\alpha=0.6$ was chosen. The results for this condition are shown in Table 1, where it can be seen that differences among the stiffness parameters are rather significant and the sensitivity with respect to the stiffness for the lower modes is so low.

According to the procedure in Fig.3, a shaft of diameter 30mm, length 375mm was installed onto the same bearing. Reduction of the shaft length resulted in the decrease of the stiffness parameter and increase of the sensitivity with respect to the stiffness parameter as shown in Fig 2b. The location parameter was chosen initially as 0.9 so that the natural frequencies of the system may be well separated. The results for this condition are shown in Fig.5 and Table 2. From Table 2, it can be seen that the sensitivities with respect to the location parameter for 3rd mode was so high, and the differences among the stiffness parameter were more significant than cases of $\ell=750$mm, while the sensitivities with respect to the stiffness parameter were improved.

Subsequently, the location parameter α for each mode. The results for this condition are shown in Fig.6 and Table 3. As shown in Table 3, differences among the stiffness parameters were much smaller than those of the previous experiments, the sensitivities with respect to the stiffness parameter were

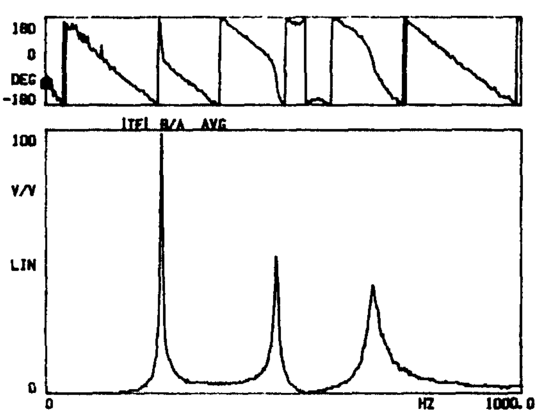

Fig. 4 Transfer fuinction from experiment When $\ell=750$mm $R=15$mm $\alpha=0.6$

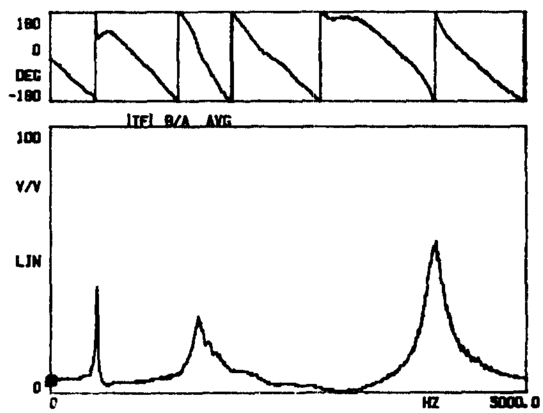

Fig. 5 Transfer function from experiment When $\ell=375$mm R=15mm $\alpha=0.9$

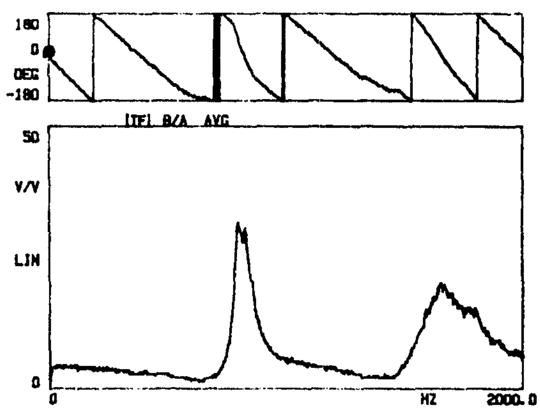

Fig. 6 Transfer function from experiment When $\ell=375$mm $R=15$mm $\alpha=0.4$

Table 2 Identified joint stiffness at $\ell=375$mm, $\alpha=0.9$

Natural frequency	Eigen value	Stiffness parameter	Bearing stiffness	$\Delta\nu/\Delta k$	$\Delta\nu/\Delta k$	W_i
495 Hz	11.24	441	7.0×10^{7}	3.9×10^{-3}	-22.13	0.361
1769 Hz	40.16	712	1.1×10^{8}	1.3×10^{-2}	-59.02	0.442
4045 Hz	91.82	3450	5.5×10^{8}	1.1×10^{-2}	-114.06	0.197
Weighted stiffness parameter 1156						
Weighted bearing stiffness 1.8×10^{8} N/m						

Table 3 Identified joint stiffness at $\ell=375$mm, $\alpha=0.4$

Natural frequency	Eigen value	Stiffness parameter	Bearing stiffness	$\Delta\nu/\Delta k$	$\Delta\nu/\Delta k$	W_i
788 Hz	17.89	619	9.8×10^7	8.7×10^{-3}	40.95	0.290
837 Hz	19.00	614	9.7×10^7	3.1×10^{-3}	33.65	0.124
1625 Hz	36.89	517	8.2×10^7	2.9×10^{-2}	−66.91	0.586

Weighted stiffness parameter 558
Weighted bearing stiffness 8.8×10^7 N/m

relatively high, and the sensitivities with respect to the location parameter were low. The weighted average stiffnes was 8.8×10^7 N/m.

5. CONCLUDING REMARKS

A method has been proposed for the identification of the bearing stiffness, which basically uses simulation charts of the eigen frequencies and their sensitiviteres with respect to the system parameters for a nondimensionalized shaft-bearing system. The proposed method yields very reliable results because

(1) measurements are made only on the natural frequencies,

(2) experimental set-up is arranged so that sensitivites with respect to the system parameters are optimized.

(3) and estimates of the stiffness are obtained in such a way that they may be consistent regardless of the dynamic mode chosen.

Through a case sutdy, the usefulness of the proposed procedure was illustrated and it was found that a great care is required especially in selecting the location of the bearings along the shaft.

In this study, however, an objective way of decision making for satisfaction or not and the reasons why the estimates of stiffness from different dynamic modes where inconsistent were not discussed in a concrete manner, which are left for further study.

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APPENDIX

Substituting Eq.(9)into Eq.(7)yields

$$u_1(z,\ \tau)=[A_{i1}\cos\Lambda_1 z+A_{i2}\sin\Lambda_1 z+A_{i3}\cosh\Lambda_2 z+A_{i4}\sinh\Lambda_2 z]e^{\nu\tau} \qquad \text{(A1)}$$

wherer the subscript i in A_{ij} indicates the specified region shown in Fig.1. Substituting Eq.($A1$) in to Eq.(6) yields. Eq.(11) and $\{A\}$, $[C]$are given by :

$$\{A\}^T=\{A_{11}A_{12}A_{13}A_{14}A_{21}A_{22}A_{23}A_{24}A_{31}A_{32}A_{33}A_{34}\} \qquad \text{(A2)}$$

$$\begin{aligned}
[C]=[&[-\Lambda_1^2\cos\Lambda_1 z_0-\Lambda_1^2\sin\Lambda_1 z_0\Lambda_2^2\cosh\Lambda_2^2 z_0\Lambda_2^2\sinh\Lambda_2 z_0\\
&0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]\\
&[(\Lambda_1^3\sin\Lambda_1 z_0-r^2\nu^2\Lambda_1\sin\Lambda_1 z_0+\omega\nu\Lambda_1^2\cos\Lambda_1 z_0)\\
&(-\Lambda_1^3\cos\Lambda_1 z_0+r^2\nu^2\Lambda_1\cos\Lambda_1 z_0+\omega\nu\Lambda_1^2\sin\Lambda_1 z_0)\\
&(\Lambda_2^3\sinh\Lambda_2 z_0+r^2\nu^2\Lambda_2\sinh\Lambda_2 z_0-\omega\nu\Lambda_2^2\cosh\Lambda_2 z_0)\\
&(\Lambda_2^3\cosh\Lambda_2 z_0+r^2\nu^2\Lambda_2\cosh\Lambda_2 z_0-\omega\nu\Lambda_2^2\sinh\Lambda_1 z_0) \qquad \text{(A3)}\\
&0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]\\
&[\cos\Lambda_1 z_1\quad \sin\Lambda_1 z_1\quad \cosh\Lambda_2 z_1\quad \sinh\Lambda_2 z_1\\
&-\cos\Lambda_1 z_1\quad -\sin\Lambda_1 z_1\quad -\cosh\Lambda_2 z_1\quad -\sinh\Lambda_2 z_1\\
&0\ \ 0\ \ 0\ \ 0]\\
&[-\Lambda_1\sin\Lambda_1 z_1\quad \Lambda_1\cos\Lambda_1 z_1\quad \Lambda_2\sinh\Lambda_2 z_1\quad \Lambda_2\cosh\Lambda_2 z_1\\
&\Lambda_1\sin\Lambda_1 z_1\quad -\Lambda_1\cos\Lambda_1 z_1\quad -\Lambda_2\sinh\Lambda_2 z_1\quad -\Lambda_2\cosh\Lambda_2 z_1\\
&0\ \ 0\ \ 0\ \ 0]\\
&[-\Lambda_1^2\cos\Lambda_1 z_1\quad -\Lambda_1^2\sin\Lambda z_1\quad \Lambda_2^2\cosh\Lambda_2 z_1\quad \Lambda_2^2\sinh\Lambda_2 z_1\\
&\Lambda_1^2\cos\Lambda_1 z_1\quad \Lambda_1^2\sin\Lambda_1 z_1\quad -\Lambda_2^2\cosh\Lambda_2 z_1\quad -\Lambda_2^2\sinh\Lambda_2 z_1\\
&0\ \ 0\ \ 0\ \ 0]\\
&[\Lambda_1^3\sin\Lambda_1 z_1-\mathrm{k}\cos\Lambda_1 z_1\qquad -A_1^3\cos\Lambda_1 z_1-\mathrm{k}\sin\Lambda_1 z_1\\
&\Lambda_2^3\sinh\Lambda_2 z_1-\mathrm{k}\cosh\Lambda_2 z_1\qquad -\Lambda_2^3\cosh\Lambda_2 z_1-\mathrm{k}\sinh\Lambda_2 z_1
\end{aligned}$$

$-\Lambda_1^3\sin\Lambda_1 z_1 \quad \Lambda_1^3\cos\Lambda_1 z_1 \quad -\Lambda_2^3\sinh\Lambda_2 z_1 \quad \Lambda_2^3\cosh\Lambda_2 z_1$

$0 \quad 0 \quad 0 \quad 0]$

$[0 \quad 0 \quad 0 \quad 0$

$\cos\Lambda_1 z_2 \quad \sin\Lambda_1 z_2 \quad \cosh\Lambda_2 z_2 \quad \sinh\Lambda_2 z_2$

$-\cos\Lambda_1 z_2 \quad -\sin\Lambda_1 z_2 \quad -\cosh\Lambda_2 z_2 \quad -\sinh\Lambda_2 z_2]$

$[0 \quad 0 \quad 0 \quad 0$

$-\Lambda_1{}^3\sin\Lambda_1 z_2 \quad \Lambda_1\cos\Lambda_1 z_2 \quad \Lambda_2\sinh\Lambda_2 z_2 \quad \Lambda_2\cosh\Lambda_2 z_2$

$\Lambda_1\sin\Lambda_1 z_2 \quad -\Lambda_1\cos\Lambda_1 z_2 \quad -\Lambda_2\sinh\Lambda_2 z_2 \quad -\Lambda_2\cosh\Lambda_2 z_2]$

$[0 \quad 0 \quad 0 \quad 0$

$-\Lambda_1^2\cosh\Lambda_1 z_2 \quad -\Lambda_1^2\sin\Lambda_1 z_2 \quad \Lambda_2^2\cosh\Lambda_2 z_2 \quad \Lambda_2^2\sinh\Lambda_2 z_2$

$\Lambda_1^2\cos\Lambda_1 z_2 \quad \Lambda_1^2\sin\Lambda_1 z_2 \quad -\Lambda_2^2\cosh\Lambda_2 z_2 \quad -\Lambda_2^2\sinh\Lambda_2 z_2$

$[0 \quad 0 \quad 0 \quad 0$

$\Lambda_1^3\sin\Lambda_1 z_2 - \mathrm{k}\cos\Lambda_1 z_2 \quad -\Lambda_1^3\cos\Lambda_1 z_2 - \mathrm{k}\sin\Lambda_1 z_2$

$\Lambda_2^3\sinh\Lambda_2 z_2 - \mathrm{k}\cosh\Lambda_2 z_2 \quad \Lambda_2^3\cosh\Lambda_2 z_2 - \mathrm{k}\sinh\Lambda_2 z_2$

$-\Lambda_1^3\sin\Lambda_1 z_2 \quad \Lambda_1^3\cos\Lambda_1 z_2 \quad -\Lambda_2^3\sinh\Lambda_2 z_2 \quad \Lambda_2^3\cosh\Lambda_2 z_2]$

$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$

$-\Lambda_1^2\cos\Lambda_1 z_3 \quad -\Lambda_1^2\sin\Lambda_1 z_3 \quad \Lambda_2^2\cosh\Lambda_2 z_3 \quad \Lambda_2^2\sinh\Lambda_2 z_3]$

$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$

$(\Lambda_1^3\sin A_1 z_3 - r^2\nu^2\Lambda_1\sin\Lambda_1 z_3 + \omega\nu\Lambda_1^2\cos\Lambda_1 z_3)$

$(-\Lambda_1^3\cos A_1 z_3 + r^2\nu^2\Lambda_1\cos\Lambda_1 z_3 + \omega\nu\Lambda_1^2\sin\Lambda_1 z_3)$

$(\Lambda_2^3\sinh A_2 z_3 + r^2\nu^2\Lambda_2\sinh\Lambda_2 z_3 - \omega\nu\Lambda_2^2\cosh\Lambda_2 z_3)$

$(\Lambda_2^3\cosh A_2 z_3 + r^2\nu^2\Lambda_2\cosh\Lambda_2 z_3 - \omega\nu\Lambda_2^2\sinh\Lambda_1 z_3)]]$